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COMPARISON SENSITIVITY DESIGN OF OUTPUT FEEDBACK SYSTEMS USING --ETC(U)
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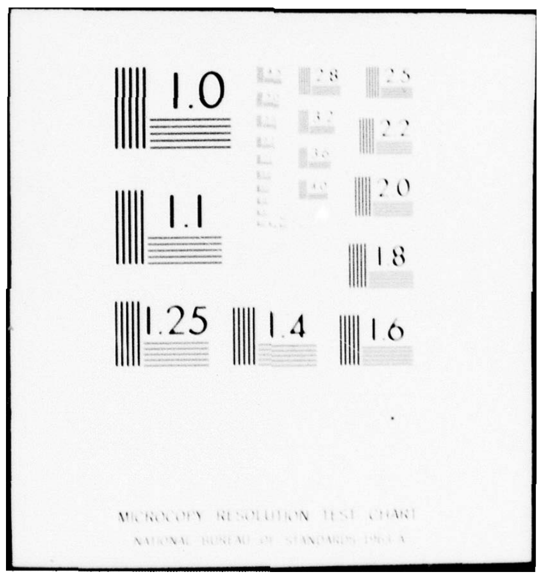
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COMPARISON SENSITIVITY DESIGN OF OUTPUT FEEDBACK SYSTEMS USING STATE OBSERVERS

BRUCE HARVEY KROGH

REPORT R-807 UILU-ENG 78-2201

UNIVERSITY OF ILLINOIS - URBANA, ILLINOIS

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A desirable property of any control design is that it be insensitive to small variations in the parameters of the controlled plant. This is necessary for several reasons. For instance, the mathematical model can only approximate the physical problem so that the assumed values of parameters for the design may be different from the actual parameter values upon implementation. Also, most systems suffer from some forms of unmeasurable or unpredictable variations due to the degeneration of physical components and adverse environmental effects. Hence, there is a need for general design methods

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20. ABSTRACT (continued)

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In this thesis the reduction of sensitivity of feedback systems to parameter variations in the plant is investigated. The concept of comparison sensitivity is used as an indication of the sensitivity performance of a given design. In particular, the question of sensitivity reduction in feedback systems which use state observers for dynamic compensation is considered leading to a design procedure which guarantees sensitivity reduction with respect to a particular comparison sensitivity reduction criterion.

The problem has been solved for the full state feedback case, and it was recently solved for the output feedback case using dynamic compensators. In both instances it is shown that sensitivity reduction is directly related to some optimal control law. A short overview of these past results is given in the sequel followed by a presentation of the results developed using state observers in the compensator dynamics. All systems discussed are assumed to be linear time invariant (LTI) systems which are state controllable and state observable.

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COMPARISON SENSITIVITY DESIGN OF OUTPUT FEEDBACK SYSTEMS USING STATE OBSERVERS

BY

BRUCE HARVEY KROGH
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THESIS

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1. INTRODUCTION

A desirable property of any control design is that it be insensitive to small variations in the parameters of the controlled plant. This is necessary for several reasons. For instance, the mathematical model can only approximate the physical problem so that the assumed values of parameters for the design may be different from the actual parameter values upon implementation. Also, most systems suffer from some forms of unmeasurable or unpredictable variations due to the degeneration of physical components and adverse environmental effects. Hence, there is a need for general design methods which guarantee some degree of insensitivity.

In this thesis the reduction of *sensitivity of feedback systems* to parameter variations in the plant is investigated. The concept of comparison sensitivity is used as an indication of the sensitivity performance of a given design. In particular, the question of sensitivity reduction in feedback systems which use state observers for dynamic compensation is considered leading to a design procedure which guarantees sensitivity reduction with respect to a particular comparison sensitivity reduction criterion.

The problem has been solved for the full state feedback case [1,2], and it was recently solved for the output feedback case using dynamic compensators [3]. In both instances it is shown that sensitivity reduction is directly related to some optimal control law. A short overview of these past results is given in the sequel followed by a

presentation of the results developed using state observers in the compensator dynamics. All systems discussed are assumed to be linear time invariant (LTI) systems which are state controllable and state observable.

2. COMPARISON SENSITIVITY WITH FULL STATE FEEDBACK

2.1. Definitions

Comparison sensitivity relates the sensitivity of one control scheme to the sensitivity of another nominally equivalent control to identical parameter variations in the controlled plant. The potential benefits of using state feedback to improve sensitivity can be evaluated by comparing the sensitivity of the closed-loop design to a nominally equivalent open-loop control. The development of these concepts has lead to the definition of the comparison sensitivity operator [4] which directly relates the open-loop and closed-loop sensitivities, and a sensitivity reduction criterion giving sufficient conditions for a particular feedback control law to guarantee sensitivity reduction in comparison to the open-loop control [1].

To make the above discussion precise, consider an LTI dynamic system described by the nth order state equations

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x^o \\ x(t) &\in R^n, \quad u(t) \in R^m. \end{aligned} \tag{1}$$

If a state feedback law is given by

$$u(t) = Fx(t), \qquad F \in R^{m \times n} \tag{2}$$

then the nominal state trajectory is given by

$$x_n(t) = e^{(A+BF)t} x^o. \tag{3}$$

Now, consider differentially small variations in the parameters of the coefficient matrices, denoted by δA, δB, and δC. Neglecting second

and higher order terms, the variation of the state trajectory, $\delta x(t)$, is given as the solution to

$$\delta \dot{x}(t) = \delta A x(t) + A\delta x(t) + \delta B u(t) + B\delta u(t), \quad \delta x(0) = 0. \tag{4}$$

Defining the open-loop state trajectory variation as

$$\delta x_o(t) = \delta x(t)\Big|_{(4)\text{ with } \delta u(t)=0} \tag{5}$$

and the closed-loop state trajectory variation as

$$\delta x_c(t) = \delta x(t)\Big|_{(4)\text{ with } \delta u(t)=F\delta x(t)} \tag{6}$$

leads to the following definition.

<u>Definition</u> (Comparison Sensitivity). Given the system (1), the feedback law (2) defines the nominal state trajectory (3). With (5) and (6) defining the open-loop and closed-loop state trajectory variations, the closed-loop system is said to be less sensitive than the nominally equivalent open-loop system with respect to a weighting matrix Z, if

$$\int_0^{t_1} \delta x_c'(t) Z \delta x_c(t) dt \le \int_0^{t_1} \delta x_o'(t) Z \delta x_o(t) dt \tag{7}$$

for all $t_1 \ge 0$. The matrix Z in (7) is a positive semi-definite symmetric $n \times n$ matrix.

The comparison sensitivity matrix is a frequency domain operator defined as follows. For the open-loop and closed-loop systems defined above, it can be shown [2] that in the frequency domain

$$\delta X_c(s) = S(s)\delta X_o(s) \tag{8}$$

where $\delta X_c(s)$ and $\delta X_o(s)$ are the Laplace transforms of $\delta x_c(t)$ and $\delta x_o(t)$,

respectively and

$$S(s) \triangleq [I - (sI-A)^{-1}BF]^{-1}. \tag{9}$$

Another useful form for S(s) in (9) is given by

$$S(s) = I + (sI-A-BF)^{-1}BF. \tag{10}$$

The form (10) is derived from (9) by applying the matrix identity $(I+UV)^{-1} = I-U(I+VU)^{-1}V$.

2.2. State Feedback Results

Having defined comparison sensitivity, the following results give a direct relationship between comparison sensitivity reduction and an optimal control which minimizes a particular quadratic performance index.

Theorem 1. [2] A sufficient condition for (7) to hold for all $t_1 \geq 0$ is that for all $\omega \in R$

$$S'(-j\omega)ZS(j\omega) - Z \leq 0 \tag{11}$$

where $S(j\omega) = S(s)|_{s=j\omega}$ defined in (9).

If the performance index

$$J = \frac{1}{2}\int_0^\infty [x'(t)Qx(t) + u'(t)Ru(t)]dt \quad \text{where } Q = Q' \geq 0 \quad \text{and} \quad R = R' > 0 \tag{12}$$

is defined, it is well known that the state feedback law which minimizes J is given by

$$u(t) = -R^{-1}B'Kx(t) \tag{13}$$

where K is the unique positive definite symmetric solution to the Riccati

equation

$$KA + A'K + Q - KBR^{-1}B'K = 0.$$

The result for state feedback of primary interest for this research may now be stated.

Theorem 2. [1] For a system described by (1) the sensitivity reduction criterion (11) is satisfied for a feedback law (2) if there exists some Q and R as in (12) such that

$$F = -R^{-1}B'K \quad \text{and} \quad Z = F'BF. \tag{14}$$

In other words, any optimal feedback control law as in (13) reduces sensitivity in the sense of comparison sensitivity for a weighting matrix given by (14).

It should be noted that Z depends integrally upon the quadratic performance index which is optimized and is therefore not at the direct disposal of the designer. Furthermore, because of the matrix factor R in (14), the rank of Z is bounded above by m, the number of inputs.

3. COMPARISON SENSITIVITY WITH OUTPUT FEEDBACK

As in other areas of linear system theory, assuming a linear combination of the states rather than all of the states is available for measurement greatly complicates the feedback control sensitivity problem. The discussion and example which follow reveal some of the reasons why the output feedback case demands theory beyond what has been presented for the state feedback case. Some recent results which deal with this more general situation are stated and these results form the basis for the application of state observers, which is the contribution of this research.

3.1. The Need for Added Dynamics

It is well known that a feedback law which is strictly a linear transformation of the outputs of a linear time invariant system will not, in general, be sufficient to place the poles of the system arbitrarily. Hence, reasonable control is feasible only when some dynamic compensation is included in the control scheme. The added dynamics may be in the forward path, as in the case of PID control, or in the feedback loop, as in the case of the state observer implementation.

Following the development of the state feedback case in the previous section, it would be reasonable to pose the comparison sensitivity problem in terms of the outputs rather than the states and to determine a sufficient condition under which the closed-loop system is less sensitive than the nominally equivalent open-loop system. The following example shows that even for a simple case, attempting to satisfy such an output sensitivity

criterion fails for conventional design procedures. The subsequent theory presented in Section 3.3 establishes that any added dynamics become inherently involved in the formulation and solution of the sensitivity reduction problem.

3.2. A Counterexample Using State Observers

Consider the following conventional design procedure. For an unstable plant, an optimal state feedback control is found using the algebraic Riccati equation. By Theorem 2, such a control guarantees sensitivity reduction with respect to a particular weighting matrix. The state feedback is implemented by direct feedback of any measurable states and the remaining unmeasurable states are estimated by a full order observer.

As an attempt to extend the sensitivity results for state feedback to the output feedback case, an output sensitivity reduction criterion is defined. Using the above design procedure, the observer dynamics are designed to satisfy the sensitivity criterion. For the following particular case, such a design fails to result in a stable observer.

For the controllable and observable LTI system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad x(t) = x^{o} \\ y(t) &= Cx(t) \end{aligned} \tag{15}$$

where

$$A = \begin{bmatrix} 0 & -2 \\ -1 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad C = [0 \quad 1]$$

the state feedback control which minimizes

$$J = \frac{1}{2}\int_{0}^{\infty}[y^2(t) + u^2(t)]dt \tag{16}$$

is given by

$$u(t) = Fx(t) \quad \text{where } F = [2.88 \quad -10.18]. \tag{17}$$

Following the approach to the output sensitivity problem suggested above, suppose the sensitivity reduction criterion to be satisfied is given by

$$\int_{0}^{t_1} \delta y_c^2(t)dt \leq \int_{0}^{t_1} \delta y_o^2(t)dt \tag{18}$$

for all $t_1 \geq 0$. The second state will be fed back directly and the first state will be estimated using a full order state observer of the form

$$\dot{\hat{x}} = [A + LC]\hat{x} + Bu - Ly \tag{19}$$

where A, B, and C are defined by the system (15) and L is the observer gain,

$$L = \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} \tag{20}$$

which is chosen to satisfy (18) and to stabilize (19).

It can be shown that a necessary and sufficient condition for (18) to hold for this design is (see Appendix)

$$\left| \frac{\begin{matrix} s^4 - (8.88+\ell_2)s^3 + (28.04-1.88\ell_1+5.88\ell_2)s^2 - (25.44-5.64\ell_1+6.64\ell_2)s \\ -24.8+3.76\ell_1-5.76\ell_2 \end{matrix}}{\begin{matrix} s^4 + (1.30-\ell_2)s^3 - (12.68-\ell_1+4.30\ell_2)s^2 - (15.26-4.30\ell_1+2.22\ell_2)s \\ -4.44+2.22\ell_1 \end{matrix}} \right| \leq 1 \tag{21}$$

where $s = j\omega$, for all $\omega \in R$

which is true if and only if

$$\begin{aligned}&[\omega^4-(28.04-1.88\ell_1+5.88\ell_2)\omega^2-24.8+3.76\ell_1-5.76\ell_2]^2+[(8.88+\ell_2)\omega^3\\&-(25.44-5.64\ell_1+6.64\ell_2)\omega]^2-[\omega^4+(12.68-\ell_1+4.30\ell_2)\omega^2-4.44+2.22\ell_1]^2\\&-[(1.30-\ell_2)\omega^3+(15.26-4.30\ell_1+2.22\ell_2)\omega]^2\le 0.\end{aligned}\tag{22}$$

Expanding these terms and collecting the coefficients of the powers of ω leads to the highest power of ω being six with the coefficient

$$-4.28+5.76\ell_1.\tag{23}$$

Hence, a necessary condition for (22) to hold is

$$\ell_1\le .74.\tag{24}$$

However, stability of the observer requires that

$$\ell_1>2.\tag{25}$$

Therefore, the conditions (24), and (25) imply that no observer implementation will satisfy both the sensitivity criterion and the stabilization requirement.

This simple example shows that added dynamics lead to sensitivity problems if the output sensitivity criterion (18) is to be satisfied. In the following section a general theory for comparison sensitivity with output feedback will be presented which shows that the added dynamics of a compensator must be included in any comparison sensitivity reduction criterion.

3.3. Output Feedback Results

This section is based entirely upon the theory developed in [3].

To study comparison sensitivity for the output feedback case the sensitivities of all variables involved in the feedback law must be

included in the sensitivity criterion. Hence, when states from a dynamic compensator as well as the process outputs are fed back, both the plant outputs and the compensator states must be used in the comparison sensitivity analysis.

To make this explicit, consider the LTI controllable and observable system given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad x(0) = x^{o} \\ y(t) &= Cx(t) \end{aligned} \tag{26}$$

$$x(t) \in R^{n}, \quad y(t) \in R^{\ell}, \quad u(t) = R^{m}.$$

Assume rank(B) = m and rank(C) = ℓ with $m \leq n$ and $\ell \leq n$. Suppose the control u(t) is given by

$$u(t) = Hz(t) + Ky(t), \qquad H \in R^{m \times k}, \qquad K \in R^{m \times \ell} \tag{27}$$

where z(t) is the state of the kth order dynamics of the compensator given by

$$\dot{z}(t) = Fz(t) + Gy(t) + Du(t), \qquad z(0) = z^{0}, \qquad z \in R^{k}. \tag{28}$$

The constant matrices F, G, and D are of appropriate dimensions.

The system (26), compensator (28), and the control (27), define a composite system with states $\overline{x}(t)$ and state equations

$$\begin{aligned} \dot{\overline{x}}(t) &= \overline{A}\,\overline{x}(t) + \overline{B}u(t) \qquad \overline{x}(0) = \begin{bmatrix} z^{o} \\ x^{o} \end{bmatrix} \\ \overline{y}(t) &= \overline{C}\,\overline{x}(t) \\ u(t) &= \overline{K}\,\overline{y}(t) \end{aligned} \tag{29}$$

where

$$\overline{A} = \begin{bmatrix} F & GC \\ 0 & A \end{bmatrix} \qquad \overline{B} = \begin{bmatrix} D \\ B \end{bmatrix}$$

$$\overline{C} = \begin{bmatrix} I_k & 0 \\ 0 & C \end{bmatrix} \qquad \overline{K} = [H \quad K].$$

For differential variations of the matrix coefficients denoted by $\delta\overline{A}$, $\delta\overline{B}$, and $\delta\overline{C}$, the resulting first order variations of the state and output variables satisfy

$$\delta\dot{\overline{x}}(t) = \delta\overline{A}\,\overline{x}(t) + \overline{A}\delta\overline{x}(t) + \delta\overline{B}u(t) + \overline{B}\delta u(t), \qquad \delta\overline{x}(0) = 0. \tag{30}$$

In a manner analogous to the state feedback case, the open-loop output trajectory variation is defined as

$$\delta\overline{y}_o(t) = \delta\overline{y}(t)\Big|_{(30)\text{ with } \delta u(t) = 0} \tag{31}$$

and the closed-loop output trajectory variation is defined as

$$\delta\overline{y}_c(t) = \delta\overline{y}(t)\Big|_{(30)\text{ with } \delta u(t) = \overline{K}\delta\overline{y}(t)}. \tag{32}$$

The comparison sensitivity criterion [4] may now be defined. Given the composite system (29) with input $u(t) = \overline{K}\,\overline{y}(t)$ defining the nominal output trajectory, the closed-loop system is less sensitive than the nominally equivalent open-loop system with respect to a positive semi-definite symmetric $(n+k)\times(n+k)$ weighting matrix Z, if for all $t_1 \geq 0$

$$\int_0^{t_1} \delta\overline{y}_c'(t)Z\delta\overline{y}_c(t)dt \leq \int_0^{t_1} \delta\overline{y}_o'(t)Z\delta\overline{y}_o(t)dt. \tag{33}$$

In a manner analogous to the state feedback case [4,2], a sufficient condition for (33) to hold is given in Theorem 3.

<u>Theorem 3</u>. [4] The plant and compensator defining the composite system (29) satisfy the condition for comparison sensitivity reduction (33) if for all ω in R

$$[I+\bar{C}(-j\omega I-\bar{A}-\bar{B}\bar{K}\bar{C})^{-1}\bar{B}\bar{K}]'Z[I+\bar{C}(j\omega I-\bar{A}-\bar{B}\bar{K}\bar{C})^{-1}\bar{B}\bar{K}]-Z \leq 0. \tag{34}$$

Letting (34) define the comparison sensitivity reduction criterion, the following theorem parallels the result of Theorem 2 for the state feedback case by relating the satisfaction of (34) to the solution of an optimal control problem.

<u>Theorem 4</u>. [3] For a plant and compensator given by the composite system (29) with $m \leq k+\ell$, a particular parameterization of the compensator (28) and (27) satisfies the sensitivity reduction criterion (34) for a positive semi-definite symmetric weighting matrix Z with rank(Z) = m if and only if the following three conditions are satisfied

$$\mathrm{rank}(\bar{K}) = m \tag{35}$$

$$Z = \bar{K}'R\bar{K} \tag{36}$$

$$\bar{K}\bar{C} = -R^{-1}\bar{B}'P \tag{37}$$

where P is the positive definite symmetric solution of the n+k order Riccati equation

$$P\bar{A}+\bar{A}'P+Q-P\bar{B}R^{-1}\bar{B}'P = 0$$

for some positive definite symmetric matrices Q and R of appropriate dimensions.

The conditions given by (36) and (37) may be viewed as a direct extension of the state feedback result since $\bar{K}\bar{C}$ is actually the state feedback matrix for the composite system (29), and (37) demands that it satisfy an optimal control law.

In [3], the conditions of Theorem 4 are applied to a canonical form for the composite system which leads to a design procedure for constructing a dynamic compensator which satisfies the sensitivity reduction criterion. The details of these results will not be discussed here, but those points which are used in the development of a design procedure using observers will be presented briefly.

Lemma 1. (Plant Canonical Form). Given a plant described by (26) with $\mathrm{rank}(CB) = m_1$, there exists nonsingular matrices T, $n\times n$, and U, $m\times m$, such that

$$CT = [I_\ell \quad 0] \tag{38}$$

and

$$T^{-1}BU = \begin{bmatrix} B_1^* \\ \hdashline B_2^* \end{bmatrix} \tag{39}$$

where

$$B_1^* = \begin{bmatrix} I_{m_1} & 0 \\ 0 & 0 \end{bmatrix} \in R^{\ell\times m} \quad , \; B_2^* = \begin{bmatrix} 0 & I_{m-m_1} \\ 0 & 0 \end{bmatrix} \in R^{(n-\ell)\times m}.$$

This lemma clearly follows by construction and implies that there are bases for the state and input spaces such that (38) and (39) hold.

It is shown in [3] that a necessary condition for $\mathrm{rank}(Z) = m$ is that $\mathrm{rank}(\bar{C}\bar{B}) = m$. By selecting D in the compensator to satisfy this rank

condition it can be shown that there exist nonsingular matrices T_s, $(k+n)\times(k+n)$, and T_o, $(k+\ell)\times(k+\ell)$, such that the composite system can be transformed to what will be called the composite system canonical form given by

$$\hat{A} \triangleq T_s\bar{A}T_s^{-1}$$
$$\hat{B} \triangleq T_s\bar{B} = \begin{bmatrix} I_m \\ 0 \end{bmatrix} \tag{40}$$
$$\hat{C} \triangleq T_o\bar{C}T_s^{-1} = [I_{k+\ell} \quad 0].$$

With $(\hat{A},\hat{B},\hat{C})$ defined above, the following equivalent conditions for sensitivity reduction can be established.

Theorem 5. [3] For the composite system (29) with the plant matrices in plant canonical form the conditions (35), (36), and (37) of Theorem 4 are equivalent to

$$\operatorname{rank}(\bar{C}\,\bar{B}) = m \tag{41}$$

and the eigenvalues of

$$\hat{A}_{22} + \hat{A}_{21}W[I_{k+\ell-m} \quad 0], \qquad W \in R^{m\times(k+\ell-m)} \tag{42}$$

have negative real parts, where $\hat{A}$ is any matrix similar to $\bar{A}$ defined in (40) and $\hat{A}$ is partitioned as

$$\hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}, \quad \hat{A}_{11} \in R^{m\times m}, \ \hat{A}_{22} \in R^{(n+k-m)\times(n+k-m)}. \tag{43}$$

The proof of this theorem given in [3] is constructive in that it leads to the following design procedure. A dynamic order k is selected for the compensator and the compensator is designed with D, F, and G as in (28) and satisfying (41). The composite system is transformed to the composite system canonical form and a matrix W is sought to satisfy condition (42) of Theorem 5. If no such matrix exists, the order k is increased until such a W can be found. It is shown in [3] that for some $k \leq n-\ell+m-m_1$ there can be constructed a compensator and W to satisfy (42).

Once W has been found, the following matrices are defined

$$\tilde{W} = \begin{bmatrix} I_m & W[I_{k+\ell-m} \quad 0] \\ 0 & I_{k+n-m} \end{bmatrix} \tag{44}$$

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \triangleq \tilde{W}^{-1}\hat{A}\tilde{W}. \tag{45}$$

A positive definite symmetric matrix Q, $(k+n-m)\times(k+n-m)$, is selected and the Lyapunov equation

$$P\tilde{A}_{22} + \tilde{A}_{22}'P + Q = 0 \tag{46}$$

is solved for P. Since, by (45) $\tilde{A}_{22}$ is equal to the matrix (42), the matrix P is a positive definite matrix uniquely solving (46). Now, an $m\times m$ positive definite matrix R is selected such that

$$R^* \triangleq R^{-1} - \tilde{A}_{12}Q^{-1}\tilde{A}_{12}' > 0 \tag{47}$$

and a $(k+n-m)\times(k+n-m)$ positive semi-definite symmetric matrix V is chosen such that

$$Q^* \triangleq V + \tilde{A}_{21}^T P Q^{-1} P \tilde{A}_{21} > 0. \tag{48}$$

The Riccati equation

$$P^* A^* + A^* P^* - P^* R^* P^* + Q^* = 0 \tag{49}$$

where

$$A^* = \tilde{A}_{11} + \tilde{A}_{12} Q^{-1} P \tilde{A}_{21}$$

is then solved for the unique positive definite symmetric matrix P^* (such a solution is guaranteed by the selection of V and R). The composite system is then stabilized and the sensitivity criterion (34) is satisfied if $\overline{K}$ is given as

$$\overline{K} = -R^{-1} P^* [I \quad -W] T_o. \tag{50}$$

As was mentioned above the details and analysis of this design procedure are given in [3]. The problem of selecting the dynamic order of the compensator, k, and the matrices D, F, and G is solved in the next section by implementing state observers as the compensator dynamics. This leads to a comprehensive design procedure and some simplification of the interpretation of the algorithm through the selection of the transformation matrices T_o and T_s.

4. COMPARISON SENSITIVITY WITH STATE OBSERVERS

4.1. Why State Observers?

The results presented in the last section are a solution to the problem of comparison sensitivity reduction when output feedback is used. It is not a totally satisfactory solution, however, because although the sensitivity criterion is satisfied and the system is stabilized by the compensator, there is no explicit form for the dynamics of the compensator and there is no direct control over the eigenvalues of the overall system.

It is reasonable to ask how known dynamic compensation techniques fit into the sensitivity design specifications. Hence, the purpose of this research was to study the sensitivity characteristics of feedback systems which use state observers for the compensator dynamics. The use of state observers is desirable since their structure is well known and the design of observers is easily computerized. More importantly, it is a property of systems which have state observers that the observer poles are also poles of the overall feedback system and determine the speed with which the error of the state estimation diminishes. Finally, since many feedback laws are given in terms of state feedback and the states often have physical significance, it is desirable to have available an estimate of the unmeasurable states to implement a state feedback scheme and to monitor the performance of the system.

4.2. The Full Order Observer

The compensator dynamics (28), if chosen to be a full order observer (i.e. $k = n$), will have the form

$$\begin{aligned} F &= A + LC \\ G &= -L \\ D &= B \end{aligned} \tag{51}$$

where the $n \times \ell$ matrix L is chosen such that the eigenvalues of F will have negative real parts. Usually the eigenvalues of the observer are chosen to be much faster than the remaining feedback system eigenvalues so that the estimation error goes to zero relatively quickly.

Assume the system (26) is in the plant canonical form given in Lemma 1. It is clear that

$$\text{rank } (\bar{C}\,\bar{B}) = \text{rank} \begin{bmatrix} B \\ CB \end{bmatrix} = m \tag{52}$$

so that the rank condition (41) of Theorem 5 is satisfied.

Now partition the matrix A as in Figure 1 and define T_o and T_s as in Figure 2. It is simple to verify that the nonsingular matrices T_o and T_s will transform the composite system (29) for the plant (26) and observer defined in (51) to composite system canonical form. The explicit form for $\hat{A}$ (40) in this case is given in Figure 3. With $\hat{A}$ partitioned as in (43), the stability of the matrix (42) is shown in Figure 4 to depend upon the stability of the observer and the existence of a matrix W which stabilizes the pair

$$A = \begin{bmatrix} A_{11} & \cdots & A_{14} \\ \vdots & & \vdots \\ A_{41} & \cdots & A_{44} \end{bmatrix} \qquad \begin{aligned} &A_{11} \in R^{m_1 \times m_1} \\ &A_{22} \in R^{(\ell-m_1)\times(\ell-m_1)} \\ &A_{33} \in R^{(m-m_1)\times(m-m_1)} \\ &A_{44} \in R^{(n-\ell+m_1-m)\times(n-\ell+m_1-m)} \end{aligned}$$

Figure 1. Partition of Plant Matrix, A.

$$T_o = \begin{bmatrix} 0 & 0 & 0 & 0 & I_{m_1} & 0 \\ 0 & 0 & I_{m-m_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n-\ell-m+m_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{m-m_1} \\ I_{m_1} & 0 & 0 & 0 & -I_{m_1} & 0 \\ 0 & I_{\ell-m_1} & 0 & 0 & 0 & -I_{\ell-m_1} \end{bmatrix}$$

$$T_s = \left[\begin{array}{ccc|c} & T_o & & 0 \\ \hline 0_{(n-\ell)\times\ell} & I_{n-\ell} & 0_{(n-\ell)\times\ell} & -I_{n-\ell} \end{array}\right]$$

Figure 2. T_o and T_s for Full Order Observer Case.

$$\hat{A} = \begin{bmatrix} A_{11} & A_{13} & A_{14} & A_{12} & 0 & 0 & -A_{13} & -A_{14} \\ A_{31} & A_{33} & A_{34} & A_{32} & A_{31}+L_{31} & A_{32}+L_{32} & 0 & 0 \\ A_{41} & A_{43} & A_{44} & A_{42} & A_{41}+L_{41} & A_{42}+L_{42} & 0 & 0 \\ A_{21} & A_{23} & A_{24} & A_{22} & 0 & 0 & -A_{23} & -A_{24} \\ 0 & 0 & 0 & 0 & A_{11}+L_{11} & A_{21}+L_{21} & A_{13} & A_{14} \\ 0 & 0 & 0 & 0 & A_{21}+L_{21} & A_{22}+L_{22} & A_{23} & A_{24} \\ 0 & 0 & 0 & 0 & A_{31}+L_{31} & A_{32}+L_{32} & A_{33} & A_{34} \\ 0 & 0 & 0 & 0 & A_{41}+L_{41} & A_{42}+L_{42} & A_{43} & A_{44} \end{bmatrix}$$

Figure 3. $\hat{A} = T_s \bar{A} T_s^{-1}$ for T_s in Figure 2.

$$\hat{A}_{22} + \hat{A}_{21} W[I_{k+\ell-m} \quad 0] = \left[\begin{array}{c|c} \begin{bmatrix} A_{44} & A_{42} \\ A_{24} & A_{22} \end{bmatrix} + \begin{bmatrix} A_{41} & A_{43} \\ A_{21} & A_{23} \end{bmatrix} \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} & X \\ \hline 0 & A + LC \end{array}\right]$$

where

$$X = \left[\begin{array}{cc|c|c} A_{41}+L_{41}+A_{41}W_{13}+A_{43}W_{23} & A_{42}+L_{42}+A_{41}W_{14}+A_{43}W_{24} & 0 & 0 \\ \hline A_{21}W_{13}+A_{23}W_{23} & A_{21}W_{14}+A_{23}W_{24} & -A_{23} & -A_{24} \end{array}\right]$$

and

$$W = \begin{array}{c} \\ m_1 \\ m-m_1 \end{array} \begin{array}{c} \begin{array}{cccc} n-m-\ell+m_1 & \ell-m_1 & m_1 & \ell-m_1 \end{array} \\ \left[\begin{array}{c|c|c|c} W_{11} & W_{12} & W_{13} & W_{14} \\ \hline W_{21} & W_{22} & W_{23} & W_{24} \end{array}\right] \end{array}$$

Figure 4. $\hat{A}_{22} + \hat{A}_{21} W[I_{k+\ell-m} \quad 0]$ for $\hat{A}$ in Figure 3.

$$\left(\begin{bmatrix} A_{44} & A_{42} \\ A_{24} & A_{22} \end{bmatrix}, \begin{bmatrix} A_{41} & A_{43} \\ A_{21} & A_{23} \end{bmatrix} \right). \tag{53}$$

The existence of such a mateix W is guaranteed by the following lemma.

Lemma 2. Given the controllable pair (A,B) where B is of the form given in (39), if A is partitioned as in Figure 1, then the pair (53) is controllable.

Proof. Let λ_i, $i = 1,2,\ldots,n$ be the n eigenvalues of A. The pair (A,B) is controllable if and only if [5]

$$\operatorname{rank}[A-\lambda_i I \vdots B] = n \quad \text{for } i = 1,2,\ldots,n. \tag{54}$$

Since B is of the form in (39), (54) implies

$$\operatorname{rank}\left[\begin{array}{cccc|cc} A_{11}-\lambda_i I & A_{12} & A_{13} & A_{14} & I_{m_1} & 0 \\ A_{21} & A_{22}-\lambda_i I & A_{23} & A_{24} & 0 & 0 \\ A_{31} & A_{32} & A_{33}-\lambda_i I & A_{34} & 0 & I_{m-m_1} \\ A_{41} & A_{42} & A_{43} & A_{44}-\lambda_i I & 0 & 0 \end{array}\right] = n \tag{54a}$$

which is true if and only if

$$\operatorname{rank}\begin{bmatrix} A_{21} & A_{22}-\lambda_i I & A_{23} & A_{24} \\ A_{41} & A_{42} & A_{43} & A_{44}-\lambda_i I \end{bmatrix} = n-m. \tag{55}$$

Rearrangement of the matrices in (55) gives the controllability of the pair (53).

From Figure 4 and Lemma 2 it is clear that there exists a matrix W such that condition (42) of Theorem 5 is satisfied for any stable full order observer.

Therefore, the following design procedure will lead to a dynamic compensator which will guarantee comparison sensitivity reduction with respect to some weighting matrix Z of rank m. Select a full order state observer (51) and choose W so that the pair (53) is stabilized, then follow the design algorithm given in the previous section. The design algorithm will be given explicitly in Section 5.1.

One important property of dynamically compensated systems which have full order state observers in the feedback loop is that the eigenvalues of the observer are also eigenvalues of the overall feedback system. This can be demonstrated by applying the similarity transformation

$$T = \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix} \tag{56}$$

to the composite system which has full order observer dynamics

$$\dot{\bar{x}}(t) = \bar{A}_f \bar{x}(t) \qquad \text{where } \bar{A}_f = \begin{bmatrix} A+LC+BH & -LC+BH \\ BH & A+BK \end{bmatrix} \tag{57}$$

for arbitrary output feedback

$$u(t) = \bar{K}\bar{y}(t) = Hz(t) + KCx(t). \tag{58}$$

The similarity transformation (56) gives

$$T\bar{A}_f T^{-1} = \begin{bmatrix} A+LC & 0 \\ BH & A+B(KC+H) \end{bmatrix}. \tag{59}$$

Note also that the other n eigenvalues of the system depend only upon the selection of $\bar{K}$, the output feedback law.

4.3. The Reduced Order Observer

Again assume that the plant matrices (A,B,C) are in plant canonical form. The compensator dynamics (28) if chosen to be a reduced order observer (i.e., $k = n-\ell$) have the form

$$\begin{aligned} F &= A_{22}^* + LA_{12}^* \\ G &= -FL + A_{21}^* + LA_{11}^* \\ D &= B_2^* + LB_1^* \end{aligned} \tag{60}$$

where

$$A = \left[\begin{array}{c|c} A_{11}^* & A_{12}^* \\ \hline A_{21}^* & A_{22}^* \end{array}\right] = \left[\begin{array}{cc|cc} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ \hline A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{array}\right]$$

B_1^*, B_2^* are defined in (39) and

$$L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}, \quad \begin{aligned} L_{11} &\in R^{(m-m_1)\times m_1} \\ L_{22} &\in R^{(k+m_1-m)\times(\ell-m_1)} \end{aligned}.$$

L is chosen such that F has eigenvalues with negative real parts.

Clearly, with the dynamics of the compensator defined by (60) we have

$$\operatorname{rank}(\bar{C}\,\bar{B}) = \operatorname{rank}\begin{bmatrix} D \\ CB \end{bmatrix} = m \tag{61}$$

so (41) in Theorem 5 is satisfied.

Now partition A as in Figure 1 and define T_o and T_s as in Figure 5. It can be verified that T_o and T_s as given will transform the composite system to the composite system canonical form. $\hat{A}$ is given

in Figure 6 and Figure 7 shows that the stabilization criterion (42) in Theorem 5 is satisfied for any stable reduced order observer and W chosen to stabilize the pair (53). Hence, by Lemma 2, a matrix W exists as required for any choice of the observer dynamics.

The design procedure using reduced order observers is identical to that using full order observers except that W is entirely determined by stabilizing the pair (53). Also, the property of the observer eigenvalues being eigenvalues of the composite feedback system holds. This is shown by writing the composite feedback as

$$\dot{\bar{x}}(t) = \bar{A}_f \bar{x}(t), \qquad \bar{x}(t) = \begin{bmatrix} z(t) \\ x_1(t) \\ x_2(t) \end{bmatrix}, \; x_1(t) \in R^{\ell} \tag{62}$$

where

$$\bar{A}_f = \begin{bmatrix} F+DH & G+DH & 0 \\ B_1^* H & A_{11}^* + B_1^* H & A_{12}^* \\ B_2^* H & A_{21}^* + B_2^* H & A_{22}^* \end{bmatrix}$$

and applying the similarity transformation

$$T = \begin{bmatrix} I & -L & -I \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \tag{63}$$

which gives

$$T\bar{A}_f T^{-1} = \begin{bmatrix} F & 0 \\ BH & A+B[K+HL \vdots H] \end{bmatrix}. \tag{64}$$

From (64) it is noted that, unlike the full order observer case, the remaining n eigenvalues of the feedback system depend explicitly upon the observer matrix L.

$$T_o = \begin{bmatrix} 0 & 0 & I_{m_1} & 0 \\ I_{m-m_1} & 0 & -L_{11} & -L_{12} \\ 0 & I_{k+m_1-m} & -L_{21} & -L_{22} \\ 0 & 0 & 0 & I_{\ell-m_1} \end{bmatrix}$$

$$T_s = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{T_o} & 0 \\ \hline -I_k & L & I_k \end{array}\right]$$

Figure 5. T_o and T_s for Reduced Order Observer Case.

$$\hat{A} = \begin{bmatrix} A_{11} & A_{13} & A_{14} & A_{12} & A_{13} & A_{14} \\ A_{31} & A_{33} & A_{34} & A_{22} & -L_{11}A_{13}-L_{12}A_{23} & -L_{11}A_{14}-L_{12}A_{24} \\ A_{41} & A_{43} & A_{44} & A_{42} & -L_{21}A_{13}-L_{22}A_{23} & -L_{21}A_{14}-L_{22}A_{24} \\ A_{21} & A_{23} & A_{24} & A_{22} & A_{23} & A_{24} \\ 0 & 0 & 0 & 0 & A_{33}+L_{11}A_{13}+L_{12}A_{23} & A_{34}+L_{11}A_{14}+L_{12}A_{24} \\ 0 & 0 & 0 & 0 & A_{43}+L_{21}A_{13}+L_{22}A_{23} & A_{44}+L_{21}A_{14}+L_{22}A_{24} \end{bmatrix}$$

Figure 6. $\hat{A} = T_s\bar{A}T_s^{-1}$ for T_s in Figure 5.

$$\hat{A}_{22} + \hat{A}_{21} W [I_{k+\ell-m} \ 0] =$$

$$\left[\begin{array}{c:cc} \begin{bmatrix} A_{44} & A_{42} \\ A_{24} & A_{22} \end{bmatrix} + \begin{bmatrix} A_{41} & A_{43} \\ A_{21} & A_{23} \end{bmatrix} \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} & \begin{matrix} -L_{21}A_{13} - L_{22}A_{23} \\ A_{23} \end{matrix} & \begin{matrix} -L_{21}A_{14} - L_{22}A_{24} \\ A_{24} \end{matrix} \\ \hdashline 0 & \multicolumn{2}{c}{A_{22}^{*} + L\, A_{12}^{*}} \end{array}\right]$$

Figure 7. $\hat{A}_{22} + \hat{A}_{21} W [I_{k+\ell-m} \ 0]$ for $\hat{A}$ in Figure 6.

4.4. Some Corollaries

The results of the preceding two sections can be summarized as a corollary to Theorem 5.

Corollary 1. Given an LTI system described by (26) with coefficient matrices (A,B,C) in plant canonic form, for arbitrary state observer dynamics of full order (51) or reduced order (60) there exist output feedback gains, $\overline{K}$, such that the composite system (29) is stable and insensitive to parameter variations in the plant with respect to a comparison sensitivity weighting matrix Z given by (36) with $u(t) = \overline{K}\,\overline{y}(t)$. Furthermore, the eigenvalues of the observer are eigenvalues of the composite feedback system.

In [3] it is shown that there exists a dynamic compensator with order $k \leq n-\ell+m-m_1$ such that the sensitivity reduction criterion is satisfied. The results for the reduced order observer give the following improvement on the upper bound of the dynamic order of the compensator needed to satisfy the sensitivity reduction criterion.

Corollary 2. There exists a dynamic compensator which satisfies the conditions of Theorem 5 for some dynamic order $k \leq n-\ell$.

Proof. It has been shown that the conditions of Theorem 5 can be satisfied using a reduced order observer as the dynamics for which $k = n-\ell$.

The particular choices of the transformation T_o and T_s for both full and reduced order observer cases lead to a useful interpretation of the sensitivity weighting matrix. Writing out (36) using the form of $\overline{K}$ given in (50) gives

$$Z = T_o' H T_o \tag{65}$$

where
$$H = \begin{bmatrix} P^*R^{-1}P^* & -P^*R^{-1}P^*W \\ -W'P^*R^{-1}P^* & W'P^*R^{-1}P^*W \end{bmatrix}.$$

Consider the full order observer case with T_o as in Figure 2. If the output $\bar{y}$ is decomposed as

$$\bar{y} = [z_1' \; z_2' \; z_3' \; z_4' \; y_1' \; y_2'] \tag{66}$$

$$z_1 \in R^{m_1}, \quad z_2 \in R^{\ell - m_1}, \quad z_3 \in R^{m-m_1}$$

$$z_4 \in R^{n-\ell+m_1-m}, \quad y_1 \in R^{m_1}, \quad y_2 \in R^{\ell-m_1}$$

then

$$T_o\bar{y} = [y_1' \; z_3' \; z_4' \; y_2' \vdots z_1' - y_1' \vdots z_2' - y_2']'. \tag{67}$$

Since the states of the observer are estimates of the states of the plant and y(t) is the first ℓ states of the plant when C is given by (38), the last two components of the vector in (67) are the error of the estimation of the state. Also, the second and third components are the estimates of the $n-\ell$ unknown states.

Similarly for the reduced order observer case with T_o as in Figure 5, and $\bar{y}$ decomposed as

$$\bar{y} = [z_1' \; z_2' \; y_1' \; y_2']', \quad z_1 \in R^{m-m_1}, \quad z_2 \in R^{n-\ell+m_1-m}, \tag{68}$$
$$y_1 \in R^{m_1}, \quad y_2 \in R^{\ell - m_1}$$

then

$$T_o\bar{y} = [y_1 \vdots z_1 - L_{11}y_1 - L_{12}y_2 \vdots z_2 - L_{21}y_1 - L_{22}y_2 \vdots y_2]. \tag{69}$$

The design of the reduced order observer gives the estimate of the $n-\ell$ unmeasurable states by $z-Ly$ where y is the ℓ measurable states. Hence, (69) shows that $T_o\bar{y}$ gives the measurable states and the estimation of the unmeasurable states.

Since the states of a system often have physical significance, the following corollary which summarizes the above discussion may be useful in evaluating the sensitivity reduction capabilities of a particular weighting matrix resulting from (36).

<u>Corollary 3</u>. Design of the feedback gains according to the state observer schemes defined by Corollary 1 leads to a sensitivity weighting matrix H (65) which weights the known states, estimates of the unknown states and in the full order observer case, the error of the state estimation given by (67) and (69).

5. THE DESIGN ALGORITHM

A summary of the design procedure described in the discussion following Theorem 5 and extended by the use of state observers in the preceding section will be presented in the following. An interactive computer program was written to implement the design algorithm and to enable adjustment of the arbitrary parameters iteratively to obtain the desired overall design objectives. This program is discussed briefly in conjunction with a descriptive flowchart. An example of the steps involved and comparison to a conventional design technique is given.

5.1. Flowchart and Interactive Software Description

Figure 8 is a flowchart of the design procedure and shows the points at which redesign of parameters can take place. Each step of the algorithm will be discussed briefly with the numbers referring to blocks on the flowchart.

1. The basic system parameters must be determined and the plant canonical form is assumed. Hence, before entering a general plant description, the triple (A,B,C) must be transformed to the plant canonical form. The computer accepts the A matrix and constructs the B and C matrices according to the dimension specifications

n - dynamic order of the system

m - number of inputs

ℓ - number of outputs

m_1 - rank of CB.

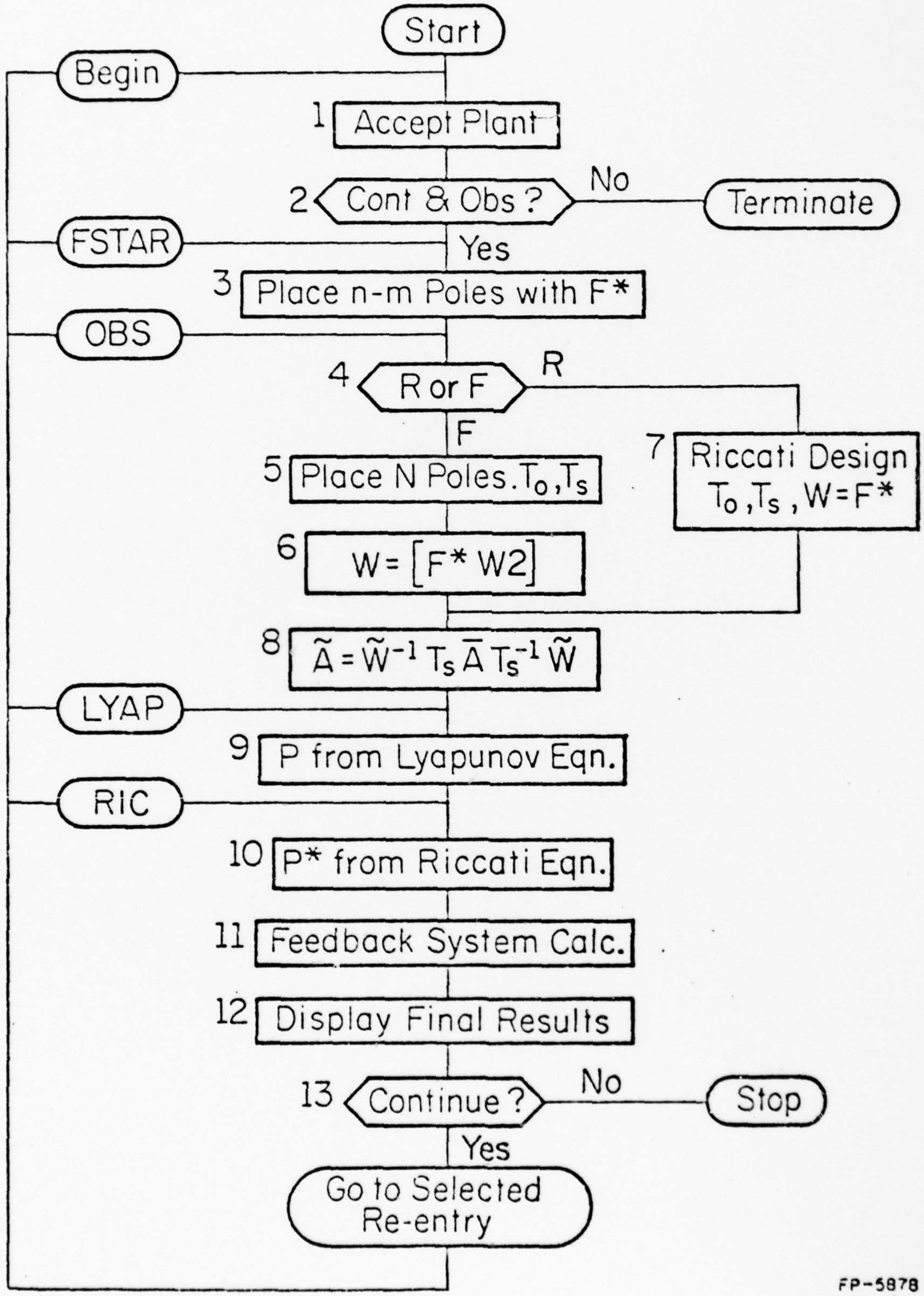

Figure 8. Design algorithm flowchart.

2. The controllability of the pair (A,B) and the observability of the pair (A,C) are tested and the program terminates if either test fails.

3. The stabilization of the pair (53) is accomplished using pole placement of the n-m poles available. The resulting feedback matrix F^* is used to determine W after the observer order has been selected.

4. Reduced or full order observer design is selected.

5. For the full order observer the matrix L is determined using pole placement to stabilize the n eigenvalues of F in (51). T_o and T_s are constructed as specified in Figure 2.

6. The matrix W2 is arbitrarily chosen by the designer and $W = [F^* \;\; W2]$ as specified in Figure 4.

7. For the reduced order observer the $n-\ell$ order dynamics are stabilized using a Riccati optimization design which requires a $(n-\ell)\times(n-\ell)$ positive definite matrix Q and an $\ell\times\ell$ positive definite matrix R. L is then found as

$$L = -KA^{*\prime}_{12}R^{-1} \tag{70}$$

where A^*_{12} is defined in (60) and K is the $(n-\ell)\times(n-\ell)$ unique positive definite symmetric solution to the algebraic Riccati equation

$$KA^{*\prime}_{22} + A^*_{22}K + Q - KA^{*\prime}_{12}R^{-1}A^*_{12}K = 0 \tag{71}$$

T_s and T_o are constructed as in Figure 5 and $W = F^*$ as specified by Figure 7 and Lemma 2.

8. Set up $\tilde{A}$ as in (45).

9. Accept Q and solve for P in the k+n-m dimensional Lyapunov equation (46).

10. Accept R^{-1} and V as in (47) and (48) and solve the m×m Riccati equation (49) for P^*.

11. The feedback gain $\bar{K}$ (50) can now be calculated as well as the sensitivity weighting matrices Z (36) and H (65). The resulting feedback system [(57) or (62)] is constructed and its eigenvalues are calculated.

12. All pertinent results of the final solution are displayed.

13. The program can be terminated or re-entered at points indicated on the flowchart to redesign some aspect of the system.

In summary, the design procedure involves the solution of the observer feedback gains (nth order or $(n-\ell)$th order), the solution of a (n+k- m) order Lyapunov equation and an mth order Riccati equation. Obviously, there are myriads of free parameters which are to be selected by the designer and it may take several design attempts to accomplish satisfactory results.

Many of the mathematical computer subroutines used were taken from the LINSYS package [6] which was also used to make the plots for the following example.

5.2. An Example

As an example of the design procedure, consider the following model of the longitudinal dynamics of an aircraft [7]. In Table 1 the seven states and three inputs are defined. The first five states come from the simplified model of the dynamics. The additional two states are

defined as integral controls to eliminate steady state error for constant set points of velocity and pitch angle. Following the design in [7], it is assumed that the velocity, pitch angle and altitude are measured directly and the pitch angle rate as well as the angle of attack must be estimated. In other words, an observer must be used to estimate the states one and four.

Using the linearized coefficients and flight conditions given in [7] the state equation in the form (26) has the coefficient matrices (A,B,C) given in Figure 9. In Table 2 certain parameters defining the flight conditions are given along with perturbed values. Assuming these perturbations, the resulting A and B matrices are given in Figure 10. The C matrix is unperturbed since the states are measured directly.

To implement the design procedure, the plant matrices (A,B,C) were transformed to the plant canonical form as in Lemma 1 by the non-singular matrices T and U in Figure 11. The transformed nominal A matrix is shown in Figure 12. Note that under the similarity transformation T, the state variables defined in Table 1 are arranged as

$$[x_2\ x_3\ x_5\ x_6\ x_7\ x_1\ x_4]'. \tag{72}$$

The design algorithm was used iteratively with the final choices of free parameters and resulting observer and feedback matrices given in Figure 13.

For a comparison of the sensitivity reduction properties of the design algorithm to a conventional design procedure using a reduced order observer, the feedback gains were calculated assuming the state feedback

Table 1. Definition of State Variables and Inputs For Aircraft Control of Longitudinal Dynamics

x_1 = angle of attack	$\dot{x}_6 = x_2$ - [Reference Velocity $\div$ 100]
x_2 = velocity $\div$ 100	$\dot{x}_7 = x_3$ - [Reference Pitch Angle]
x_3 = pitch angle	u_1 = elevator angle
x_4 = pitch angle rate	u_2 = flap angle
x_5 = altitude $\div$ 100	u_3 = throttle

Table 2. Flight Condition Parameters

Variable	Nominal Value	% of Perturbation	Perturbed Value
Center of Graviey	.20 ft	+ 5%	.21 ft
Velocity	190.66 ft/sec	+ 5%	200.19 ft/sec
Weight	4000 lbs	- 5%	3800 lbs
Air Density	.004842 slugs/ft^3	+ 5%	.005084 slugs/ft^3
Moment of Inertia	2050 slugs-ft^2	- 2%	2009 slugs-ft^2

$$A = \begin{bmatrix} -3.09 & -0.18 & 0 & 1 & 0 & 0 & 0 \\ 0.14 & -0.07 & -0.32 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -0.74 & 0.09 & 0 & -1.01 & 0 & 0 & 0 \\ -1.91 & 0 & 1.91 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & -0.24 & 0 \\ 0 & -0.04 & 0.16 \\ 0 & 0 & 0 \\ -1.37 & -1.48 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Figure 9. Nominal A, B and C Matrices for Aircraft Control Example.

$$A = \begin{bmatrix} -3.58 & -0.16 & 0 & 1 & 0 & 0 & 0 \\ 0.14 & -0.07 & -0.32 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -0.26 & -0.09 & 0 & -1.14 & 0 & 0 & 0 \\ -2.00 & 0 & 2.00 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & -0.28 & 0 \\ 0 & -.05 & 0.16 \\ 0 & 0 & 0 \\ -1.62 & -1.70 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Figure 10. Perturbed A and B Matrices for Aircraft Control Example.

$$T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} 0 & 4.44 & -0.73 \\ 0 & -4.10 & 0 \\ 6.08 & -1.03 & 0 \end{bmatrix}$$

Figure 11. Transformations for Plant Canonical Form for (A,B,C) in Figure 9.

$$A = \begin{bmatrix} -0.07 & -0.32 & 0 & 0 & 0 & 0.14 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1.91 & 0 & 0 & 0 & -1.91 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -0.18 & 0 & 0 & 0 & 0 & -3.09 & 1 \\ 0.09 & 0 & 0 & 0 & 0 & -0.74 & -1.01 \end{bmatrix}$$

Figure 12. A Matrix in Plant Canonical Form for Aircraft Control Example.

For the reduced order observer:

$$Q = \begin{bmatrix} 900 & 0 \\ 0 & 5625 \end{bmatrix} \qquad R = \mathrm{diag}[0.01,\ 0.01,\ 0.01,\ 0.01,\ 0.01]$$

$$F = \begin{bmatrix} -33.3 & 0.66 \\ -1.96 & -27.4 \end{bmatrix} \qquad D = \begin{bmatrix} -1.14 & 1 & 0 \\ -0.05 & 0 & 1 \end{bmatrix}$$

$$G = \begin{bmatrix} -37.9 & 36.8 & 524 & 0 & 0 \\ -3.39 & -722 & 48.3 & 0 & 0 \end{bmatrix}$$

Other design parameters:

$$Q = \left[\begin{array}{c|cc} I_4 & \multicolumn{2}{c}{0} \\ \hline & 1230 & 0 \\ 0 & & \\ & 0 & 900 \end{array}\right]$$

$$R = \begin{bmatrix} .515 & 0.00198 & -0.069 \\ 0.00198 & 0.149 & -0.104 \\ -0.069 & -0.104 & 0.408 \end{bmatrix}$$

$$V = 0$$

Resulting feedback gains:

$$\bar{K} = \begin{bmatrix} 0.08 & 0.06 & 2.23 & 1.69 & -1.37 & 2.14 & 0.14 \\ 4.57 & -3.23 & 5.78 & -90.3 & -72.3 & 0.74 & 1.33 \\ -0.74 & 2.89 & -0.56 & 81.7 & 10.2 & 0.12 & 2.15 \end{bmatrix}$$

Figure 13. Design Parameters and Resulting Feedback Gains for Sensitivity Design of the Aircraft Control.

was possible using the solution to the optimal control problem

$$\text{minimize } J = \frac{1}{2}\int_{0}^{\infty}[x'(t)Qx(t) + u'(t)Ru(t)]dt. \tag{73}$$

The arrangement of the state variables was as in (72). It is well known that the state feedback which solves (73) is given by

$$u(t) = -R^{-1}B'Kx(t) \tag{74}$$

where K is the nxn positive definite solution to the algebraic Riccati equation

$$KA + A^{T}K + Q - KBR^{-1}B'K = 0. \tag{75}$$

The matrices Q, R, and resulting state feedback (74) are given in Figure 14.

As is conventionally done, the state feedback matrix was decomposed into the five columns which multiply the known states and the two columns which multiply the estimated states. The resulting feedback gains used in the system with the reduced order observer used in the sensitivity design is given in Figure 15.

As an evaluation of the sensitivity properties of these two design procedures, Figures 17 to 20 show the trajectories of the states which were to converge to set points under nominal and perturbed conditions. The set points and perturbed initial conditions are given in Figure 16.

The clearest demonstration of the improved sensitivity characteristics of the sensitivity design algorithm is a comparison of Figures 18 and 20. Clearly the Riccati design is much more sensitive to the perturbations used. The solutions presented here are not intended to represent

particularly desirable solutions to the aircraft control problem, but to merely demonstrate the sensitivity design procedure under some practical design conditions.

$Q = \mathrm{diag}[1.44,\ 0.0025,\ 12.25,\ 12.25,\ 25,\ 0.0025,\ .01]$

$R = \mathrm{diag}[2,\ 2,\ 2]$

$$F_R = \begin{bmatrix} -2.31 & -0.02 & -0.14 & -2.47 & -0.10 & 0.02 & -0.04 \\ 0.02 & 3.24 & 2.46 & -0.13 & -0.40 & -1.32 & 0.53 \\ -0.04 & -5.36 & -0.27 & 0.09 & -3.51 & 0.53 & -2.21 \end{bmatrix}$$

Figure 14. Design Parameters and Resulting State Feedback Gains, F_R, for Riccati Design of the Aircraft Control.

$$\bar{K}_R = \begin{bmatrix} 0.02 & -0.04 & -2.29 & -0.96 & -0.35 & -2.47 & -1.04 \\ -1.32 & 0.53 & -1.46 & 16.91 & 22.98 & -0.13 & -0.40 \\ 0.53 & -2.21 & 0.47 & -63.53 & -4.30 & 0.09 & 3.51 \end{bmatrix}$$

Figure 15. Feedback Gains, $\bar{K}_R$, Used to Implement the Riccati Design with the Reduced Order Observer.

$$E' = [\ 0 \quad 0 \quad 0 \quad -0.1467 \quad -0.08727 \quad 0 \quad 0 \quad 0 \quad 0\ ]$$

$$\delta x(0)' = [\ 0 \quad 0 \quad -0.1 \quad -0.0436 \quad 0 \quad 0 \quad 0 \quad 0.05 \quad 0\ ]$$

Figure 16. Set Points, E, and Initial Condition Perturbations, $\delta x(0)$, for the Aircraft Control Example.

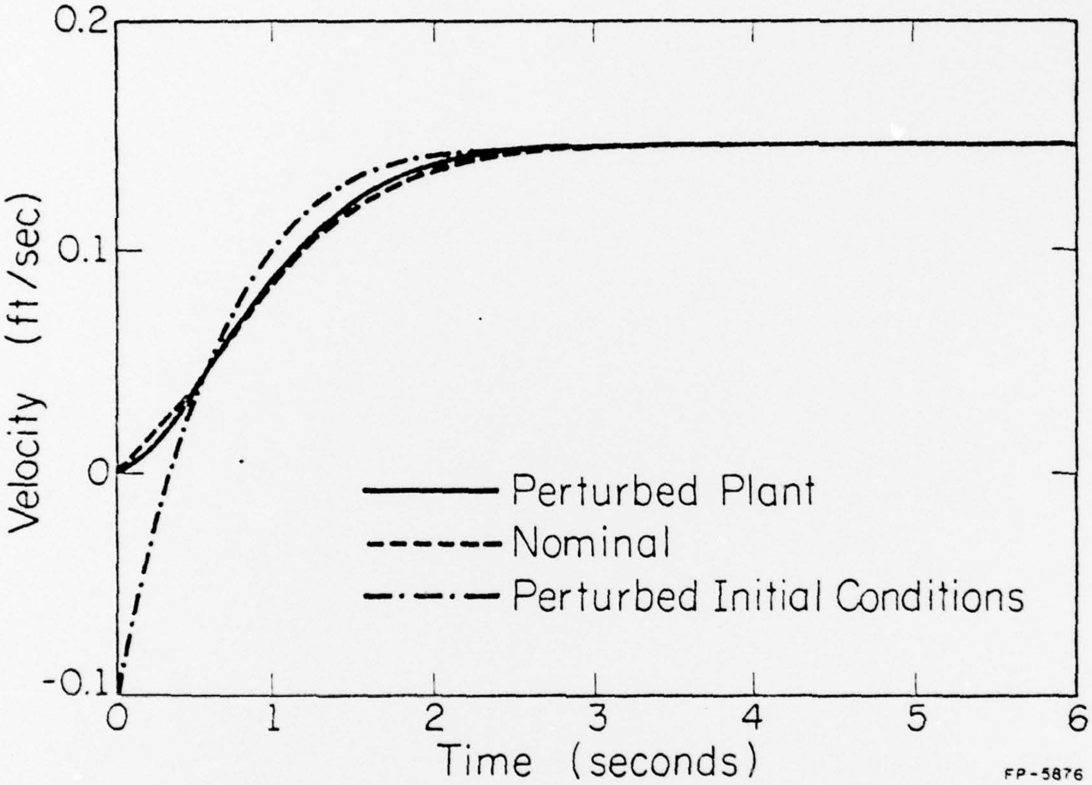

Figure 17. Velocity trajectories for sensitivity design.

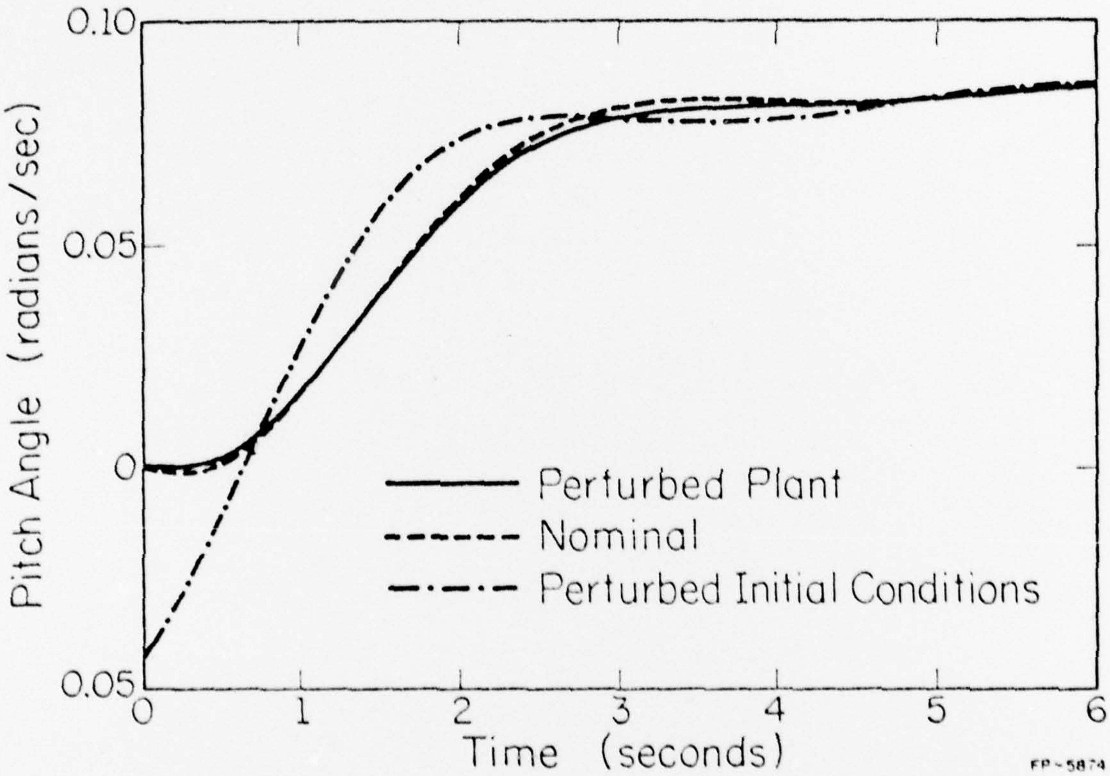

Figure 18. Pitch angle trajectories for sensitivity design.

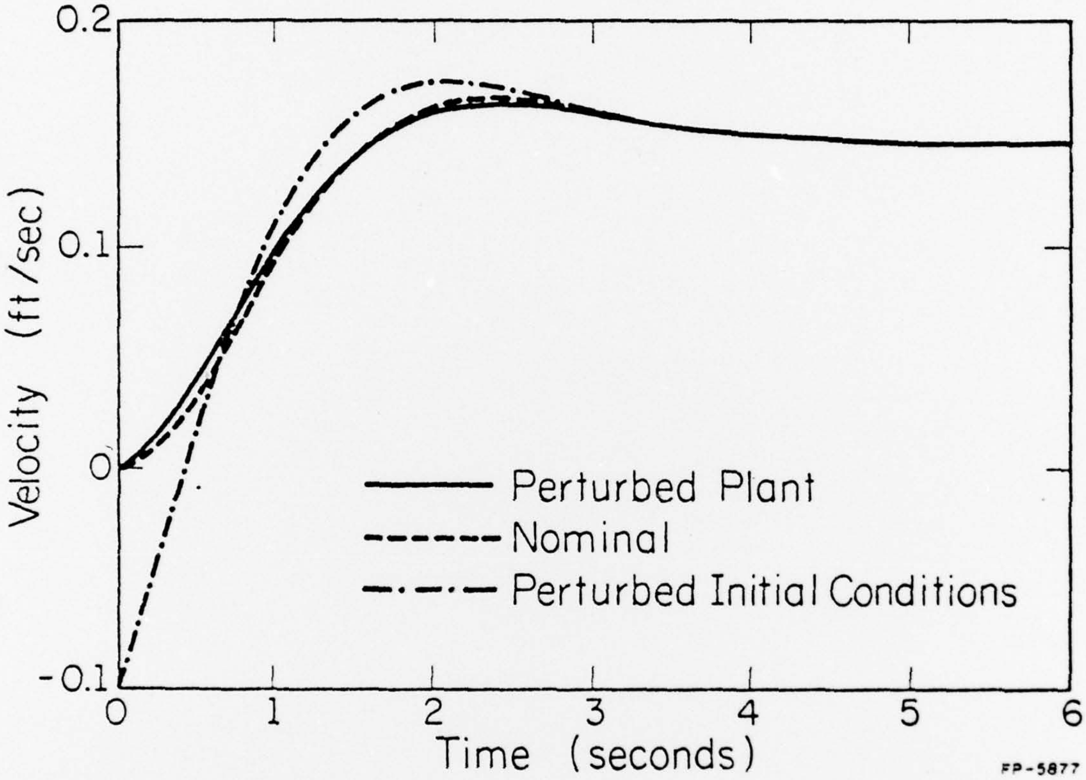

Figure 19. Velocity trajectories for Riccati design.

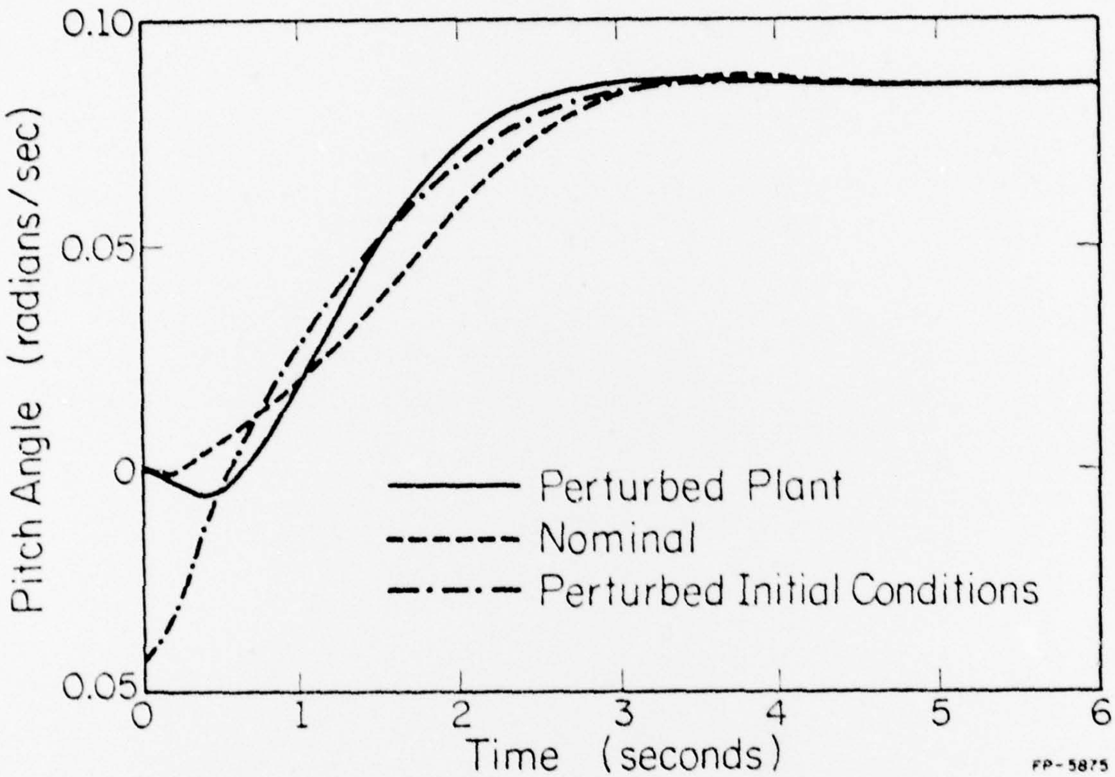

Figure 20. Pitch angle trajectories for Riccati design.

6. CONCLUSIONS

The advantageous aspects of the design of sensitivity reducing dynamic compensators through the use of state observers have been explained in the body of this thesis. Some of the shortcomings of the design procedure as it is developed here should be pointed out.

Experience bears out that it is difficult to attain desirable control objectives through the free parameters at the disposal of the designer. This is because of the complexities of the equations which lead to the final feedback gains. Although the observer design is arbitrary and determines some of the overall feedback system eigenvalues, the remaining eigenvalues are not easily placed. Consequently, the value of the algorithm could be enhanced by further research of the dependence of the final result upon the free parameters.

More insight into possible applications of the design method could be gained by perhaps considering some special cases. The sensitivity weighting matrix H (65) would be more malleable under the design constraints if some of the components in the output trajectory variations in (67) and (69) were not present. This would be the case if $\ell=m=m_1$, for instance, which would often occur in applications.

In conclusion it should be pointed out that although the design procedure appears to lead to a solution of the regulator problem, it has much wider application. Any design which calls for stabilizing compensation may have improved sensitivity characteristics by employing the algorithm developed in this thesis. For instance, the example presented in the last

section included an integral control to track a constant set point. Similarly, more complex tracking design procedures which are robust under steady state conditions and require stabilizing compensators, such as [8], may have improved transient sensitivity properties through the use of the design algorithm developed here.

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APPENDIX

DERIVATION OF THE SENSITIVITY REDUCTION CRITERION FOR THE COUNTEREXAMPLE

Consider the closed-loop configuration in Figure A.1 where all variables and transfer functions are scalars. $P_n(s)$ represents the plant transfer function for nominal values of the plant parameters. $G(s)$ and $H(s)$ are the controller transfer functions. Let $\delta Y_c(s)$ and $\delta Y_o(s)$ be the Laplace transforms of the first order output variations for the closed-loop control and the nominally equivalent open-loop control, respectively, resulting from differential changes in the plant parameters.

It is shown in [4] that

$$\delta Y_c(s) = S(s)\delta Y_o(s) \tag{A.1}$$

where $S(s)$, the comparison sensitivity function, is defined as

$$S(s) = \frac{1}{1-P_n(s)\ G(s)\ H(s)} . \tag{A.2}$$

Hence, a necessary and sufficient condition for a particular closed-loop control to be sensitivity reducing is [4]

$$|S(j\omega)| \leq 1 \quad \text{for all } \omega \in R \tag{A.3}$$

The example of Section 3.2 can be put into the configuration of Figure A.1 by defining the feedback transfer functions

$$H_1(s) = \frac{M(s)}{U(s)} = \frac{2.88s-14.40+2.88(\ell_1-\ell_2)}{s^2-(3+\ell_2)s+(\ell_1-2)} \tag{A.4}$$

and

$$H_2(s) = \frac{M(s)}{U(s)} = -10.18 + \frac{-2.88\ell_1 s+8.64\ell_1+5.76\ell_2}{s^2-(3+\ell_2)s+(\ell_1-2)} \tag{A.5}$$

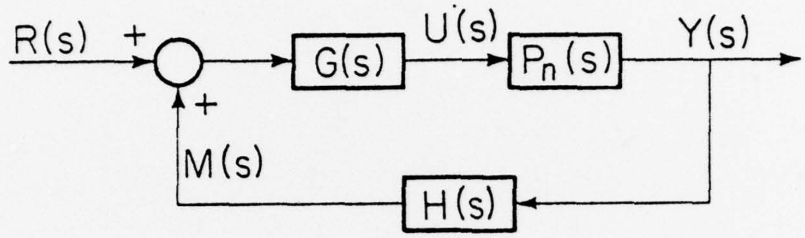

Figure A.1. General feedback control configuration.

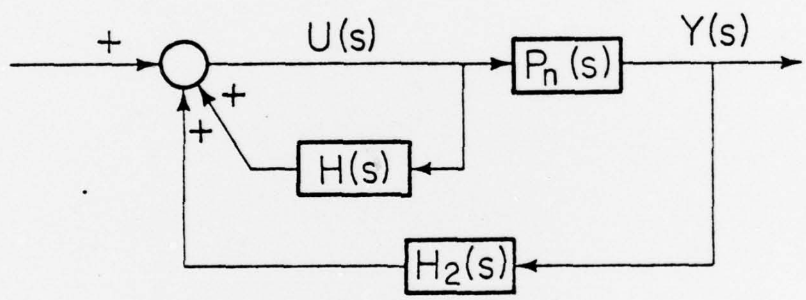

Figure A.2. Feedback control for the counterexample.

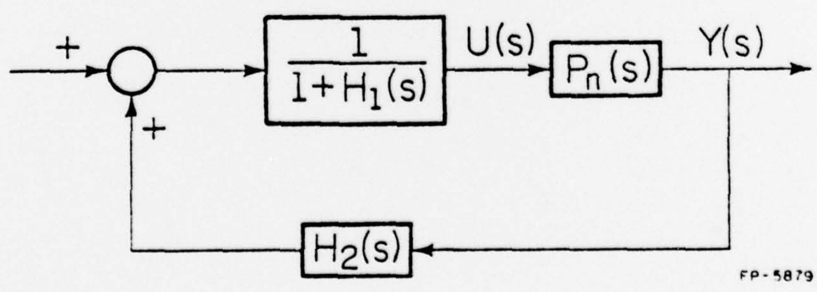

Figure A.3. Configuration equivalent to Figure A.2.

where M(s) is the Laplace transform of m(t) defined as

A.6. $$m(t) = 2.88\hat{x}_1(t) - 10.18x_2(t) \tag{A.6}$$

ℓ_1 and ℓ_2 are defined as the observer gains in equation (20). In other words, m(t) is the feedback signal resulting from $y(t) = x_2(t)$; $\hat{x}_1(t)$, the first state of the observer (19); and the optimal feedback gains (17).

Figure A.2 is a block diagram of the example in terms of $H_1(s)$, $H_2(s)$ and $P_n(s)$, where

A.7. $$P_n(s) = \frac{s-1}{s^2-3s-2} \tag{A.7}$$

Figure A.3 is an equivalent representation of the feedback system in a form similar to Figure A.1. Hence, for the example, we have from Figure A.3 and equation (A.2)

A.8. $$S(s) = \frac{1}{1 - \dfrac{P_n(s)\,H_2(s)}{1 - H_1(s)}} \tag{A.8}$$

Substituting A.4, A.5 and A.7 into A.8 gives

$$S(s) = \frac{s^4 - (8.88+\ell_2)s^3 + (28.04 - 1.88\ {}_1 + 5.88\ell_2)s^2 - (25.44 - 5.64\ell_1 + 6.64\ell_2)s - 24.8 + 3.76\ell_1 - 5.76\ell_2}{s^4 + (1.30-\ell_2)s^3 - (12.68 - \ell_1 + 4.30\ell_2)s^2 - (15.26 - 4.30\ell_1 + 2.22\ell_2)s - 4.44 + 2.22\ell_1}\,. \tag{A.9}$$

Equation A.9 in conjunction with the sensitivity reduction criterion A.3 gives the inequality (21) in section 3.2.